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Exact solution of the row-convex polygon perimeter generating function

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Abstract. An explicit expression is obtained for the perimeter generating function $G(y) = \sum_{n\geq 2} a_n y^{2n}$ for row-convex polygons on the square lattice, where a_n is the number of $2n$ step row-convex polygons. An asymptotic expression for $a_n \sim A\mu^n n^{-3/2}$ is obtained, where $\mu = 3 + 2\sqrt{2}$ and A is given. We also show that the generating function is an algebraic function and that it satisfies an inhomogeneous linear differential equation of degree three.

1. Introduction

Despite strenuous efforts over the past 40 years, the problem of 'self-avoiding polygons' in two or more dimensions remains unsolved. A large amount of numerical evidence exists and, for the hexagonal lattice, the critical point is exactly known, as is the critical exponent (Nienhuis 1982, 1984, Guttmann and Enting 1988a, Enting and Guttmann 1989). In that case, the polygon generating function for $a_n^{(\mathrm{p})}$, the number of $2n$ step polygons, is

$$G_{(\mathrm{p})}(y) = \sum_{n=2}^{\infty} a_n^{(\mathrm{p})} y^{2n} \sim A(1 - y^2/y_c^{(\mathrm{p})^2})^{2-\alpha}$$

where α is believed to be exactly $\frac{1}{2}$ for all two-dimensional lattices, $y_c^{(\mathrm{p})^2} = 2 - \sqrt{2}$ for the honeycomb lattice, and $y_c^{(\mathrm{p})^2} \approx 0.143\,680\ldots$ for the square lattice.

Simpler versions of the problem have been solved. The simplified versions are usually subsets of the self-avoiding polygon problem, and indeed are generally exponentially small subsets. Nevertheless, it is hoped that their solution will provide insight into the original problem. Further, many of the simplified problems are of interest in their own right, not just as combinatorial objects, but as problems that arise in other areas of science, such as computer science (e.g. Delest and Viennot 1984).

The first such restricted problem, where we confine ourselves to the square lattice, is that of staircase polygons, first solved by Temperley (1956) and independently by Pólya (1969). In that case, if we define the bottom left-most vertex as the origin,

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steps from the origin may only be north or east. A more difficult problem is that of convex polygons, which are polygons whose minimum bounding rectangle has the same perimeter as the polygon. This problem is of interest in computer science, and was first solved by Delest and Viennot (1984), and was subsequently independently solved by Guttmann and Enting (1988b), Lin and Chang (1988) and Kim (1988), all by different methods. Convex polygons are a superset of the staircase polygons. The staircase polygons may be considered as polygons convex with respect to a line at 45° to a lattice axis (Pólya 1969). A superset of convex polygons arises if we impose convexity in the direction of only one of the lattice axes. If the number of vertical steps in the minimum bounding rectangle equals the number of vertical steps in the polygon, but the horizontal steps are unrestricted, we speak of 'row-convex' polygons. This model was introduced by Temperley (1956), who called it 'Model Q' in his hierarchy of models. Temperley obtained an *implicit* equation for the generating function.

In this paper we obtain an explicit solution for the generating function. We show that it is an algebraic function and we give an explicit expression for the degree-four algebraic equation. We also obtain the corresponding third-order inhomogeneous linear differential equation, and an asymptotic expression for a_n, the number of $2n$ step row-convex polygons. Further work on the convex polygon problem shows it satisfies a second order algebraic equation, and a first-order inhomogeneous differential equation. These are also given explicitly.

In terms of critical behaviour, the staircase polygon generating function behaves as

$$G_{(s)}(y) = \sum_{n=2}^{\infty} a_n^{(s)} y^{2n} \sim A_{(s)}(1 - y^2/y_c^{(s)^2})^{1/2}$$

where $y_c^{(s)^2} = 1/4$, the convex polygon generating function behaves as

$$G_{(c)}(y) = \sum_{n=2}^{\infty} a_n^{(c)} y^{2n} \sim A_{(c)}(1 - y^2/y_c^{(c)^2})^{-2}$$

where $y_c^{(c)^2} = 1/4$, and the row-convex polygon generating function behaves as

$$G(y) = \sum_{n=2}^{\infty} a_n y^{2n} \sim A(1 - y^2/y_c^2)^{1/2}$$

where $y_c^2 = 3 - 2\sqrt{2}$.

In a subsequent paper we hope to describe the two-variable generating function for row-convex polygons in terms of both area and perimeter.

2. Generating function

Let g_r be the generating function for row-convex polygons whose first row contains exactly r squares. Then Temperley (1956) has shown that g_r satisfies the following set of equations

$$\begin{aligned}
g_1 &= y^4 + y^2(g_1 + 2g_2 + 3g_3 + 4g_4 + \ldots) \\
g_2 &= y^6 + y^2[2y^2 g_1 + (1 + 2y^2)g_2 + (2 + 2y^2)g_3 + (3 + 2y^2)g_4 + \ldots] \\
g_3 &= y^8 + y^2[3y^4 g_1 + (2y^2 + 2y^4)g_2 + (1 + 2y^2 + 2y^4)g_3 + (2 + 2y^2 + 2y^4)g_4 + \ldots] \\
g_4 &= y^{10} + y^2[4y^6 g_1 + (3y^4 + 2y^6)g_2 + (2y^2 + 2y^4 + 2y^6)g_3 + (1 + 2y^2 + 2y^4 + 2y^6)g_4 + \ldots]
\end{aligned} \tag{1}$$

and similarly for $r > 4$. From these it is easily shown that g_r satisfies the recurrence relation

$$g_{r+4} - 2(1 + y^2)g_{r+3} + (1 + 3y^2 + 3y^4 - y^6)g_{r+2} - 2y^2(1 + y^2)^2 g_{r+1} + y^4 g_r = 0. \tag{2}$$

Trying a solution of the form $g_r = \lambda^r$ leads to the characteristic equation

$$\left[\lambda^2 - \lambda(1 + y + y^2 - y^3) + y^2\right]\left[\lambda^2 - \lambda(1 - y + y^2 + y^3) + y^2\right] = 0 \tag{3}$$

and hence

$$g_r = \sum_{j=1}^{4} A_j \lambda_j^r \tag{4}$$

where A_j are arbitrary functions of y (but independent of r), and the λ_j are the four solutions of (3). The full perimeter generating function is given by

$$\begin{aligned}
G(y) &= \sum_{r=1}^{\infty} g_r \\
&= \sum_{j=1}^{4} A_j \frac{\lambda_j}{1 - \lambda_j}.
\end{aligned} \tag{5}$$

Because of the considerable algebraic complexity of finding the A_j this is as far as Temperley went. However, using the computer algebra program *Mathematica* (Wolfram 1988) we have been able to find an explicit expression for $G(y)$. The four roots of (3) are

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{2}\left[1 - y + y^2 + y^3 + (1 - 2y - y^2 - y^4 + 2y^5 + y^6)^{1/2}\right] \\
\lambda_2 &= \tfrac{1}{2}\left[1 - y + y^2 + y^3 - (1 - 2y - y^2 - y^4 + 2y^5 + y^6)^{1/2}\right] \\
\lambda_3 &= \tfrac{1}{2}\left[1 + y + y^2 - y^3 + (1 + 2y - y^2 - y^4 - 2y^5 + y^6)^{1/2}\right] \\
\lambda_4 &= \tfrac{1}{2}\left[1 + y + y^2 - y^3 - (1 + 2y - y^2 - y^4 - 2y^5 + y^6)^{1/2}\right].
\end{aligned} \tag{6}$$

In the limit $y \to 0$ we have

$$\begin{aligned}
\lambda_1 &\sim \mathrm{O}(1) \qquad & \lambda_2 &\sim \mathrm{O}(y^2) \\
\lambda_3 &\sim \mathrm{O}(1) \qquad & \lambda_4 &\sim \mathrm{O}(y^2)
\end{aligned} \tag{7}$$

whilst $g_r \sim O(y^{2r+2})$, thus we must have $A_1 = 0, A_3 = 0$. Let

$$H(y) = \sum_{r=1}^{\infty} r g_r \tag{8}$$

then g_r, $H(y)$ and $G(y)$ are given by

$$\begin{aligned} g_r &= A_2\lambda_2^r + A_4\lambda_4^r \\ H &= \frac{A_2\lambda_2}{(1-\lambda_2)^2} + \frac{A_4\lambda_4}{(1-\lambda_4)^2} \\ G &= \frac{A_2\lambda_2}{1-\lambda_2} + \frac{A_4\lambda_4}{1-\lambda_4}. \end{aligned} \tag{9}$$

Using equations (1), (5) and (8) enables us to write g_1 and $g_1 - g_2$ in the form

$$\begin{aligned} g_1 &= y^4 + y^2 H(y) \\ g_2 - g_1 &= y^4(y^2-1) + y^2(2y^2-1)G(y) \end{aligned} \tag{10}$$

which provide the 'initial conditions' for the recurrence relation. Substituting (9) into (10) gives two linear equations which can be solved for the two unknowns A_2 and A_4. Combining all this together, and after considerable manipulation, we obtain the following expression for the generating function $G(y)$:

$$\begin{aligned} G(y) = \frac{1}{\Delta}\Big\{ &(y^2-1)(-21+47y^2-35y^4+5y^6) - 3(y^2-1)^2(1+y^2)(1-6y^2+y^4)^{1/2} \\ &- 9\sqrt{2}(y^2-1)^2\left[(y^4-1)(y^2-1) - (y^4-1)(1-6y^2+y^4)^{1/2}\right]^{1/2} \\ &- \sqrt{2}y(y^4-1)\left[(y^4-1)(y^2-1) + (y^4-1)(1-6y^2+y^4)^{1/2}\right]^{1/2}\Big\} \end{aligned} \tag{11}$$

where

$$\Delta = 4(18 - 38y^2 + 23y^4 - 2y^6). \tag{12}$$

The generating function can be expanded in a Taylor series about $y = 0$ to give the polynomial counts which are listed in table 1 to order y^{50}. These coefficients agree with the series expansions obtained by Whittington (unpublished) and by our own work using transfer matrix techniques analogous to those described by Guttmann and Enting (1988b).

An asymptotic expansion for a_n is obtained as follows: changing the variable from y to $x = y^2$ and expanding enables (11) to be written in the form

$$d_0 + d_1(x_c - x)^{1/2} + d_2(x_c - x) + d_3(x_c - x)^{3/2} + O(x_c - x)^2 \tag{13}$$

where $d_0, \ldots, d_3$ are the following constants

$$\begin{aligned} d_0 &= 8(17 - 11\sqrt{2}) + K_1 + K_2 \\ d_1 &= 2^{-3/4}(1+\sqrt{2})(K_1 - K_2) - 24(2)^{5/4}(10 - 7\sqrt{2}) \\ d_2 &= \tfrac{1}{4}\left[(3+4\sqrt{2})K_1 - (7+3\sqrt{2})K_2\right] - 4(91 - 72\sqrt{2}) \\ d_3 &= \frac{2^{-1/4}}{16}\left[96(51\sqrt{2} - 74) + (41+31\sqrt{2})K_1 + (27-17\sqrt{2})K_2\right] \end{aligned} \tag{14}$$

Table 1. Coefficients of the row-convex polygon generating function, where a_n is the number of $2n$ step polygons embeddable on the square lattice.

$2n$	a_n
4	1
6	2
8	7
10	28
12	122
14	558
16	2 641
18	12 822
20	63 501
22	319 554
24	1 629 321
26	8 399 092
28	43 701 735
30	229 211 236
32	1 210 561 517
34	6 432 491 192
36	34 364 148 528
38	184 463 064 936
40	994 430 028 087
42	538 165 340 289 0
44	29 226 425 965 907
46	159 227 245 772 460
48	870 004 781 620 093
50	4 766 330 416 567 254
⋮	
100	23 009 883 134 845 019 582 736 684 790 483 311

where

$$K_1 = -72\sqrt{2}(676 - 478\sqrt{2})^{1/2}$$
$$K_2 = 16\sqrt{2}(1970 - 1393\sqrt{2})^{1/2}. \tag{15}$$

Then Taylor expanding $(x_c - x)^{1/2}$ and $(x_c - x)^{3/2}$ and collecting the appropriate terms gives

$$a_n \sim (3 + 2\sqrt{2})^{n-1/2} n^{-3/2} \left[c_0 + \frac{c_1}{n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right] \tag{16}$$

where

$$\begin{aligned} c_0 &= \frac{97 + 60\sqrt{2}}{(47)^2\sqrt{\pi}} \left[(1 + \sqrt{2})(52090 - 36833\sqrt{2})^{1/2} + 6(10 - 7\sqrt{2})\right] \\ &= 0.102\,834\,715\,3765\ldots \\ c_1 &= \frac{3}{2^6(47)^4\sqrt{\pi}} \left[4(47)^2(51 - 14\sqrt{2})d_3 - (512729 + 407516\sqrt{2})d_1\right] \\ &= 0.038\,343\,814\,8233\ldots. \end{aligned} \tag{17}$$

Using equation (16) gives $a_{50} = 2.299\,88 \times 10^{34}$, which should be compared with the exact value given in table 1.

3. Algebraic and differential equations

The rational exponents that appear in $G(y)$ suggest that it is an algebraic function. By suitably manipulating and 'squaring' (11) it can be shown that it satisfies the following degree-four algebraic equation

$$\begin{aligned}(-18+38y^2-23y^4+2y^6)\,G^4+(y^2-1)\left(-21+47y^2-35y^4+5y^6\right)G^3 \\ +2(y^2-1)^2\left(-4+10y^2-11y^4+2y^6\right)G^2 \\ +\left(y^2-1\right)^3\left(-1+3y^2-7y^4+y^6\right)G=y^4\left(y^2-1\right)^4 \qquad (18)\end{aligned}$$

which is plotted in figure 1. The plot shows clearly the geometrical nature of the singularities of the function, as well as its four branched nature. Series expansions are only able to 'plot' the branch passing through the origin (as they are a Taylor expansion about this point) and only for $|y| < y_c$. Furthermore, the explicit expression (11), if plotted directly, also only gives a subset of the branches. We denote the above algebraic equation by [6,8,10,12;12], where the numbers are the degrees of the polynomial coefficients, starting with the coefficient of the highest degree of G, and the semicolon delimits the inhomogeneous term.

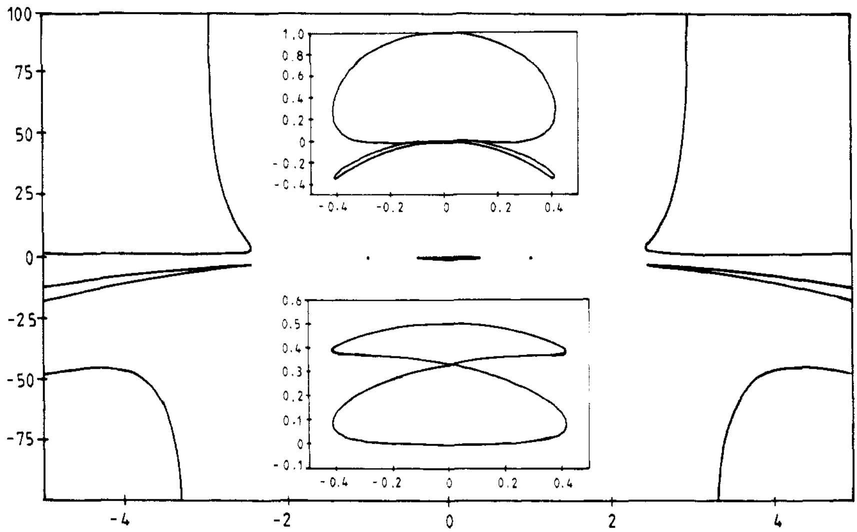

Figure 1. A plot of the real part of the (in general) four-branched algebraic generating function $G(y)$ plotted against y. Note that there are isolated points at $(\pm 1, 0)$. The upper inset is an enlargement of the origin, and the lower inset the function at infinity i.e. $1/y$ plotted against $1/G$.

The singularities of the algebraic function are given by the roots of the resultant of the algebraic equation and its partial derivative with respect to the dependent variable. Thus for the row-convex generating function the resultant is

$$16\left(y^2-1\right)^{18}y^2\left(1+y^2\right)^6\left(1-6y^2+y^4\right)^2\left(-18+38y^2-23y^4+2y^6\right) \qquad (19)$$

and hence it has 16 singularities $y=\pm 0$, $y=\pm 1$, $y=\pm \mathrm{i}$, $y=\pm 1\pm\sqrt{2}$ and the six roots of the last factor plus an additional singularity at infinity.

Using the generating function for convex polygons it is easily shown that it satisfies the following [8,14;20] degree-two algebraic equation

$$(4y^2-1)^4G_{(c)}^2+2y^4(4y^2-1)^2(-1+6y^2-11y^4+4y^6)G_{(c)}$$
$$=-y^8(16y^{12}-24y^{10}+153y^8-140y^6+58y^4-12y^2+1) \qquad (20)$$

which only has three finite singularities at $y=0,\ \pm 1/2$.

As shown by Forsyth (1902) any algebraic function of degree k satisfies a homogeneous linear differential equation of degree k, or an inhomogeneous linear differential equation of degree $k-1$. Using the method of Forsyth we obtain the following degree-three inhomogeneous differential equation for the row-convex generating function:

$$p_3(y)\frac{\mathrm{d}^3G}{\mathrm{d}y^3}+p_2(y)\frac{\mathrm{d}^2G}{\mathrm{d}y^2}+p_1(y)\frac{\mathrm{d}G}{\mathrm{d}y}+p_0(y)G=q(y) \qquad (21)$$

where

$$p_3(y)=-\left(y^4-1\right)^3\left(1-6y^2+y^4\right)^2\left(-18+38y^2-23y^4+2y^6\right)$$
$$\times\left(-8-156y^2+310y^4-14y^6+93y^8-47y^{10}-45y^{12}-3y^{14}+6y^{16}\right)$$
$$p_2(y)=-12y\left(y^2-1\right)^2\left(1+y^2\right)^3\left(1-6y^2+y^4\right)(692-2916y^2-1500y^4$$
$$+25092y^6-38449y^8+15245y^{10}-96y^{12}+7693y^{14}-7263y^{16}$$
$$-377y^{18}+2342y^{20}-809y^{22}+74y^{24})$$
$$p_1(y)=12\left(y^2-1\right)\left(1+y^2\right)(80-3424y^2+40076y^4-104196y^6-161432y^8$$
$$+1148772y^{10}-1835093y^{12}+883415y^{14}+389953y^{16}-277691y^{18}$$
$$-76739y^{20}-114089y^{22}+194609y^{24}-49889y^{26}-31144y^{28}$$
$$+19690y^{30}-3510y^{32}+196y^{34}) \qquad (22)$$
$$p_0(y)=-24y(412-4148y^2+20092y^4-90000y^6+321181y^8-1052035y^{10}$$
$$+2006120y^{12}-1998216y^{14}+969965y^{16}+667y^{18}-128962y^{20}$$
$$+74592y^{22}-180209y^{24}+135691y^{26}-15820y^{28}-28312y^{30}$$
$$+14843y^{32}-2463y^{34}+122y^{36})$$
$$q(y)=24y\left(-1+y^2\right)(144-1952y^2+7040y^4+31184y^6$$
$$-235812y^8+640300y^{10}-783798y^{12}+366108y^{14}-23573y^{16}$$
$$+29898y^{18}-11552y^{20}-70816y^{22}+62262y^{24}$$
$$-7432y^{26}-11026y^{28}+5764y^{30}-1189y^{32}+98y^{34}).$$

Analogous to the previous notation this a [42,39,38,37;37] differential equation, and it is very clear that the algebraic equation is a far more economical representation of the same function. This is an important feature of the algebraic equation particularly when approximating a function as is done in series analysis or when numerically searching for exact solutions. A change of variable from y to $x=y^2$ can be made in the differential equation with the coefficients still remaining polynomial. If this is done the equation in x becomes a [22,21,19,18;18] differential equation, and (18) becomes a [3,4,5,6;6] algebraic equation. The corresponding inhomogeneous equation for the convex polygons is of degree one, with

$$
\begin{aligned}
p_1^{(c)}(x) &= 8x^5(4x-1)^4 \\
p_0^{(c)}(x) &= -2(4x-1)^2(-1+15x-99x^2+361x^3-742x^4+770x^5-320x^6+96x^7) \\
q^{(c)}(x) &= x^2(1-6x+11x^2-4x^3)(1-15x+99x^2-361x^3+726x^4-666x^5+160x^6-96x^7)
\end{aligned}
\tag{23}
$$

and hence is a [9,9;12] equation. This contrasts with the degree-two *homogeneous* equation found by Guttmann and Enting (1988b), which is a [4,3,2;3] equation.

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